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An artificial parameter Linstedt–Poincaré method for the periodic solutions of nonlinear oscillators in which the restoring force is inversely proportional to the dependent variable

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Abstract

An artificial parameter Linstedt–Poincaré method is presented and used to determine the periodic solution of a nonlinear singular oscillator for small and large amplitudes. Comparisons with the results of harmonic balance, two-level iterative, linearized harmonic balance and parameter perturbation methods are presented.

1. Introduction

In a recent short communication published in this journal, Mickens [1] considered the following nonlinear ordinary differential equation:

$$\ddot{y} = -\frac{1}{y}, \quad y(0) = A, \quad \dot{y}(0) = 0, \tag{1}$$

where the dots denote differentiation with respect to t, and determined the frequency of oscillation by means of first- and second-order harmonic balance techniques applied to (cf. Eq. (1))

$$y\ddot{y} + 1 = 0, \quad y(0) = A, \quad \dot{y}(0) = 0, \tag{2}$$

which has the following invariant:

$$\tfrac{1}{2}\dot{y}^2 + \ln y = \ln A. \tag{3}$$

The first- and second-order harmonic balance methods yield the following approximate solutions and frequencies when applied to Eq. (2):

$$y_1(t) = A\cos(\omega_1 t), \quad \omega_1^2 = \frac{2}{A^2} \tag{4}$$

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and

$$y_2(t)=\frac{10}{9}A\left(\cos(\omega_2 t)-\frac{1}{10}\cos(3\omega_2 t)\right),\quad \omega_2^2=\frac{162}{100}\frac{1}{A^2},\tag{5}$$

respectively. Since the exact frequency of oscillation is given by $\omega_{ex}^2=\pi/2A^2$, the relative errors in the frequency corresponding to the first- and second-order harmonic balance procedure are 12.8% and 1.6%, respectively.

By means of a two-level iterative method for Eq. (2) which can be written as

$$\ddot{y}_{k+1}+\Omega_{k+1}^2 y_{k+1}=\Omega_{k+1}^2 y_k-(\ddot{y})_k^2 y_k,\quad y(0)=A,\quad \dot{y}(0)=0,\tag{6}$$

Mickens [1] found that, upon using $y_0(t)=A\cos(\theta)$ where $\theta=\Omega t$, the first iteration yields

$$y_1(t)=\frac{A}{24}(23\cos(\Omega_1 t)+\cos(3\Omega_1 t)),\quad \Omega_1^2=\frac{4}{3A^2}\tag{7}$$

and the frequency thus obtained differs from the exact one by 7.9%.

Mickens [1] also provided comparisons between the results summarized above and those obtained by means of the application of the homotopy perturbation method [2–6] to (cf. Eq. (1))

$$\ddot{y}+\omega^2 y=p(\omega^2 y-y^{-1}),\tag{8}$$

$$0\ddot{y}+1y=p\ddot{y}y\tag{9}$$

and

$$\ddot{y}+0y+p(\ddot{y})^2 y=0,\tag{10}$$

which coincide with Eq. (2) for $p=1$, and where p is a homotopy parameter which is set equal to unity at the end of the calculations.

If $y(t)$ in Eq. (8) is expanded as

$$y(t)=y_0(t)+py_1(t)+O(p^2),\tag{11}$$

the homotopy perturbation method yields the same frequency as that of Eq. (4), whereas, if $y(t)$ is expanded as in Eq. (11) and the coefficients 1 and 0 of Eq. (9) are expanded as [2–6]

$$1=\omega^2+pa_1+O(p^2),\tag{12}$$

$$0=1+pb_1+O(p^2),\tag{13}$$

respectively, then the homotopy perturbation method predicts the same frequency as that of Eq. (7) and the solution is given by

$$y(t)\approx y_0(t)+y_1(t)=A\cos(\Omega_1 t)+\frac{A^3}{32}(\cos(\Omega_1 t)-\cos(3\Omega_1 t))\tag{14}$$

and, therefore, is valid for small values of the amplitude, i.e., $A\ll 1$. Finally, if $y(t)$ is expanded as in Eq. (11) and the coefficient 0 in Eq. (10) is expanded as in Eq. (13), the homotopy perturbation method predicts the same frequency as that of Eq. (7) and the following approximate solution (cf. Eq. (11)):

$$y(t)\approx y_0(t)+y_1(t)=\frac{A}{24}(23\cos(\Omega_1 t)+\cos(3\Omega_1 t)),\tag{15}$$

after correcting a typographical error in Mickens's equation (38).

It must be pointed out that the "expansion of constants" as in Eqs. (12) and (13) was proposed by He [2–6].

Eq. (1) does not contain a small parameter and, therefore, is not readily amenable to perturbation methods based on the presence of small parameters [7]. However, it may be analyzed by methods that introduce an artificial parameter such as modified Linstedt–Poincaré techniques [2,8–12], the homotopy perturbation method [3–6,13] and linear delta expansions [14–16], by first introducing a linear stiffness term and an artificial or book-keeping parameter and then expanding both the solution and the frequency of oscillation in terms of this artificial parameter which is set to unity at the end of the calculations. In some modified Linstedt–Poincaré techniques, the frequency of oscillation is determined by a minimization procedure based on either the absolute value of the difference between the frequency of the linear oscillator and that of the real one [8] or the square of this difference [9]. In modified Linstedt–Poincaré techniques based on a linear delta series expansion, the frequency of oscillation is determined by minimizing the sum of a finite number of terms in the series expansion for the frequency of oscillation [14–16]. By way of contrast, other modified Linstedt–Poincaré techniques [10–12,2] and the homotopy perturbation method [3–6,13] do not require any minimization procedure for the determination of the frequency of oscillation.

In this paper, we first apply an artificial parameter Linstedt–Poincaré method for the determination of the periodic solution and the frequency of oscillation of Eq. (1). This technique is based on the introduction of a new dependent variable, a linear stiffness term and an artificial or book-keeping parameter, and the expansion of both the solution and the frequency of oscillation in power series of the artificial parameter which is set to unity at the end of the calculations. Therefore, the technique presented here is a modified Linstedt–Poincaré method analogous to the ones described in the previous paragraph but which does not require any minimization procedure; a similar technique has been applied by He [2,12] in his studies of the Duffing equation. We then present a comparison between the results of this technique and those of Mickens' iterative procedure [1] and the linearized harmonic balance method presented by Beléndez et al. [17].

2. Formulation 1

Eq. (1) can be written as

$$\ddot{y}+\omega^2 y=\ddot{y}+\omega^2 y-y-y^2\ddot{y}, \quad y(0)=A, \quad \dot{y}(0)=0, \tag{16}$$

which has been obtained from Eq. (1) by multiplication by y^2 and addition of inertia and a linear stiffness term.

Upon introducing $\theta=\omega t$, Eq. (16) can be written as

$$y''+y=y-\frac{1}{\omega^2}y-y^2y''+y'', \quad y(0)=A, \quad y'(0)=0, \tag{17}$$

where the prime denotes differentiation with respect to θ. Furthermore, upon introducing a book-keeping parameter p, Eq. (17) can be expressed as

$$y''+y=p\left(y-\frac{1}{\omega^2}y-y^2y''+y''\right), \quad y(0)=A, \quad y'(0)=0, \tag{18}$$

which coincides with Eq. (17) upon setting $p=1$.

By looking for solutions of Eq. (18) as

$$y(\theta)=y_0(\theta)+py_1(\theta)+O(p^2), \tag{19}$$

$$\omega^2=\omega_0^2+p\omega_1^2+O(p^2), \tag{20}$$

it is an easy exercise to show that, at $O(p^0)$,

$$y_0''+y_0=0, \quad y_0(0)=A, \quad y_0'(0)=0, \tag{21}$$

whose solution is

$$y_0(\theta)=A\cos(\theta). \tag{22}$$

At $O(p^1)$, one obtains

$$\begin{aligned} y_1'+y_1&=y_0''+y_0-\frac{y_0}{\omega_0^2}-y_0^2y_0'' \\ &=\left(-\frac{A}{\omega_0^2}+\frac{3A^3}{4}\right)\cos(\theta)+\frac{A^3}{4}\cos(3\theta), \quad y_1(0)=0, \quad y_1'(0)=0. \end{aligned} \tag{23}$$

The absence of secular terms in y_1 requires that

$$\omega_0^2 = \frac{4}{3A^2} \tag{24}$$

and the solution of Eq. (23) is then

$$y_1(\theta) = -\frac{A^3}{32}(\cos(3\theta) - \cos(\theta)). \tag{25}$$

At $O(p^2)$, one obtains

$$\begin{aligned} y_2'' + y_2 &= y_1'' + y_1 - \frac{1}{\omega_0^2}\left(y_1 - \frac{\omega_1^2}{\omega_0^2} y_0\right) - (y_0^2 y_1'' + 2y_0 y_1 y_0'') \\ &= P_1\cos(\theta) + P_3\cos(3\theta) + P_5\cos(5\theta), \quad y_2(0) = 0, \quad y_2'(0) = 0, \end{aligned} \tag{26}$$

where

$$P_1 = -\frac{A^3}{32\omega_0^2} + A\frac{\omega_1^2}{\omega_0^4} - \frac{1}{64}A^5, \tag{27}$$

$$P_3 = \frac{A^3}{32}\left(8 + \frac{1}{\omega_0^2}\right) - \frac{19}{128}A^5, \quad P_5 = -\frac{11A^5}{128} \tag{28}$$

and the absence of secular terms in $y_2(\theta)$ requires that $P_1 = 0$, i.e., $\omega_1^2 = \frac{5}{96}A^2\omega_0^2 = \frac{5}{72}$. Therefore, a two-term approximation of Eq. (20) with $p = 1$ yields

$$\omega^2 \approx \omega_0^2 + \omega_1^2 = \frac{4}{3A^2}\left(1 + \frac{5A^2}{96}\right) \tag{29}$$

and the solution of Eq. (26) is

$$y_2(\theta) = -\frac{P_3}{8}(\cos(3\theta) - \cos(\theta)) - \frac{P_5}{24}(\cos(5\theta) - \cos(\theta)), \tag{30}$$

where

$$P_3 = \frac{A^3}{4} - \frac{1}{8}A^5, \quad P_5 = -\frac{11}{128}A^5 \tag{31}$$

and, therefore, the substitution of Eqs. (22), (25) and (30) into Eq. (19) indicates that the two-term approximation to the solution is only valid for small amplitudes of oscillation.

Eq. (24) yields the same frequency of oscillation as that of the first iteration of Mickens' two-level iterative method (cf. Eq. (7)) and Eq. (22) coincides with the first approximation of the harmonic balance method (cf. Eq. (4).)

3. Formulation 2

In order to obtain solutions which are valid for large amplitudes of oscillation, Eq. (1) is first written as

$$\ddot{y} + \omega^2 y = \omega^2 y - y(\ddot{y})^2, \quad y(0) = A, \quad \dot{y}(0) = 0, \tag{32}$$

which has been obtained from Eq. (2) by multiplying that equation by $\ddot{y}$ and adding a linear stiffness term.

Eq. (32) can be written upon introducing $\theta = \omega t$ and the artificial parameter p as

$$y'' + y = p(y - \omega^2 y(y'')^2), \quad y(0) = A, \quad y'(0) = 0, \tag{33}$$

whose solution can be obtained by employing Eqs. (19) and (20) as follows.

At $O(p^0)$, one obtains Eqs. (21) and (22) while, at $O(p^1)$, it is easy to obtain

$$y_1'' + y_1 = y_0 - \omega_0^2 y_0 (y_0'')^2)$$
$$= \left(A - 3\frac{\omega_0^2 A^3}{4}\right)\cos(\theta) - \frac{\omega_0^2 A^3}{4}\cos(3\theta),$$

$$y_1(0) = 0, \quad y_1'(0) = 0 \tag{34}$$

and the absence of secular terms yields

$$y_1(\theta) = \frac{A}{24}(\cos(3\theta) - \cos(\theta)), \quad \omega_0^2 = \frac{4}{3A^2} \tag{35}$$

and, therefore, a two-term approximation to the solution (cf. Eq. (19)) is

$$y(\theta) \approx y_0(\theta) + y_1(\theta) = \frac{A}{24}(23\cos(\theta) + \cos(3\theta)) \tag{36}$$

and Eqs. (35) and (36) coincide with Eq. (7), i.e., they coincide with the frequency of oscillation and the solution provided by the first iteration of Mickens' two-level iterative technique [1] and, therefore, ω_0 differs from the exact frequency of oscillation by 7.9%.

At $O(p^2)$, one can easily obtain

$$y_2'' + y_2 = y_1 - \omega_1^2 y_0 (y_0'')^2 - \omega_0^2 y_1 (y_0'')^2 - 2\omega_0^2 y_0 y_0'' y_1''$$
$$= P_1 \cos(\theta) + P_3 \cos(3\theta) + P_5 \cos(5\theta), \quad y_2(0) = 0, \quad y_2'(0) = 0, \tag{37}$$

where

$$P_1 = -\tfrac{13}{72}A - \tfrac{3}{4}A^3\omega_1^2, \quad P_3 = -\tfrac{4}{9}A - \tfrac{1}{4}A^3\omega_1^2, \quad P_5 = -\tfrac{19}{72}A \tag{38}$$

and the absence of secular terms requires that $P_1 = 0$ and

$$y_2(\theta) = \frac{83}{1728}A(\cos(3\theta) - \cos(\theta)) - \frac{19}{1728}A(\cos(5\theta) - \cos(\theta)), \quad \omega_1^2 = -\frac{13}{54A^2}. \tag{39}$$

Therefore, a two-term approximation to the solution and the frequency of oscillation can be obtained from Eqs. (19) and (20) after setting $p = 1$ as

$$y(\theta) \approx y_0(\theta) + y_1(\theta) + y_2(\theta) = \frac{A}{1728}(1626\cos(\theta) + 83\cos(3\theta) + 19\cos(5\theta)), \tag{40}$$

$$\omega^2 \approx \omega_0^2 + \omega_1^2 = \frac{59}{54A^2}, \tag{41}$$

respectively.

The two-term approximation to the frequency of oscillation reported in Eq. (41) differs from the exact frequency of oscillation by 16.6% and is smaller than the frequency corresponding to Eq. (35) which, in turn, differs from the exact one by 7.9%. However, symbolic calculations performed with Mathematica indicate that the frequencies corresponding to the 10- and 11-term approximation provided by Eq. (20) are $\omega_{10} \approx 1.24830744/A^2$ and $\omega_{11} \approx 1.249554157/A^2$ which differ from the exact value by about 0.4% and 0.3%. Furthermore, Eq. (40) clearly indicates that, for the two-term approximation to the solution, the contribution of the third- and fifth-order harmonics is smaller than that of the fundamental frequency.

Remark 1. Eqs. (17) and (32) can be written as

$$L(y) = F(y, y''; \omega^2), \quad y(0) = A, \quad y'(0) = 0, \tag{42}$$

where

$$F(y(\theta), y''(\theta); \omega^2) = y - \frac{1}{\omega^2}y - y^2 y'' + y'' \tag{43}$$

and

$$F(y(\theta), y''(\theta); \omega^2) = y - \omega^2 y (y'')^2, \tag{44}$$

for Eqs. (17) and (32), respectively, and

$$L(y) = y'' + y, \tag{45}$$

is a linear operator. Eq. (42) can be analyzed by means of Adomian's decomposition technique [18]. Alternatively, Eq. (42) can be written, upon using the method of variation of parameters, as

$$y(\theta) = A\cos(\theta) + \int_0^\theta F(y(s), y''(s); \omega^2)\sin(\theta - s)\,\mathrm{d}s, \tag{46}$$

whose solution can be obtained by means of either iterative methods, e.g.,

$$y_{k+1}(\theta) = A\cos(\theta) + \int_0^\theta F(y_k(s), y_k''(s); \omega^2)\sin(\theta - s)\,\mathrm{d}s \tag{47}$$

or by means of homotopy by introducing an artificial parameter p as

$$y(\theta) = A\cos(\theta) + p\int_0^\theta F(y(s), y''(s); \omega^2)\sin(\theta - s)\,\mathrm{d}s, \tag{48}$$

so that Eq. (48) coincides with Eq. (47) upon setting $p = 1$.

By expanding $y(\theta)$ and ω^2 in Eq. (48) as in Eqs. (19) and (20), respectively, collecting terms in equal powers of the parameter p and using the non-secularity condition at each order in the artificial parameter, it is easily shown that the resulting approximations are identical to those presented in the two formulations above for Eqs. (43) and (44).

Remark 2. By differentiating Eq. (47) twice with respect to θ, adding Eq. (47) to the resulting equation and replacing ω by Ω_{k+1} in Eq. (44), one obtains $y_{k+1}'' + y_{k+1} = y_k - \Omega_{k+1}^2 y_k (y_k'')^2$ which coincides with Eq. (6) of Mickens' two-level iterative method upon using $\theta = \Omega_{k+1} t$.

Remark 3. By introducing a linear stiffness term and $\theta = \omega t$, Eq. (1) can be written as

$$y'' + y = y - \frac{1}{\omega^2 y}, \quad y(0) = A, \quad y'(0) = 0. \tag{49}$$

If one were to introduce an (multiplicative) artificial parameter p in the right-hand side of Eq. (49) and use Eqs. (19) and (20), one would find that, at $O(p^0)$, the solution is the same as that of Eq. (22), while, at $O(p^1)$, one would have

$$y_1'' + y_1 = y_0 - \frac{1}{\omega_0^2 y_0} = A\cos(\theta) - \frac{1}{\omega_0^2 A\cos(\theta)}, \quad y_1(0) = 0, \quad y_1'(0) = 0, \tag{50}$$

where the second term in the right-hand side is proportional to $f(\theta) = 1/\cos(\theta)$ which is neither absolutely nor square integrable in $[0, 2\pi]$. In addition, $f(\theta)$ is unbounded at $\theta = \pi/2$ and $\theta = 3\pi/2$. Therefore, the Fourier series expansion of $f(\theta)$ does not converge to $f(\theta)$ in the classical sense [19,20]. In fact, if one tried to expand it as

$$\frac{1}{\cos(\theta)} = \sum_{n=1}^{\infty} a_{2n+1}\cos((2n+1)\theta), \tag{51}$$

one would find that $a_1 = a_5 = a_9 = \cdots = 2$ and $a_3 = a_7 = a_{11} = \cdots = -2$ by either multiplying Eq. (51) by $\cos(\theta)$, using the properties of trigonometric functions and equating the coefficients of equal harmonics on the left- and right-hand sides of the resulting equation, or by simply using the well-known expressions for the determination of the Fourier series and writing $\cos((2n+1)\theta)$ in terms of powers of $\cos(\theta)$, but the resulting series does not even converge; for example, it does not converge at $\theta = 0$ where $f(0) = 1$.

If one were to ignore the above comments, proceed naively with the use of the series of Eq. (51) in Eq. (50) and eliminate secular terms, one would obtain the same value of ω_0^2 as that reported in Eq. (4) and, at $O(p^2)$, one would obtain

$$y_2'' + y_2 = y_2 + \frac{\omega_1^2}{\omega_0^4 y_0} + \frac{1}{\omega_0^2 y_0^2} y_1, \quad y_2(0) = 0, \quad y_2'(0) = 0, \tag{52}$$

whose right-hand side contains $f(\theta)$ and $g(\theta) = f^2(\theta)$, and $g(\theta)$ is neither absolutely nor square integrable in $[0, 2\pi]$ and is unbounded and more singular than $f(\theta)$ at $\theta = \pi/2$ and $\theta = 3\pi/2$. Therefore, $g(\theta)$ cannot be expanded in Fourier series in the classical sense [19,20].

The above remarks may indicate why Mickens [1] considered Eqs. (2) and (6), other authors considered Eqs. (8)–(10) and this author has considered Eqs. (18) and (32) which do not contain the singular term $1/y$ which appears in Eq. (1). It should also be noted that Eqs. (1) and (3) clearly indicate that $\ddot{y}$ and $\dot{y}$ are unbounded where $y = 0$.

Remark 4. As stated above, the Fourier series presented in Eq. (51) is divergent; however, it converges in a weak sense to $f(\theta)$ [21,22]. This can be shown by using the identity

$$\sec(\theta) + \sec(\theta)\cos(2n\theta) = 2(\cos(\theta) - \cos(3\theta) + \cdots + (-1)^{n+1}\cos((2n-1)\theta), \tag{53}$$

which can be obtained by multiplying the right-hand side of this expression by $\cos(\theta)$ and simplifying, using the cosine product-to-sum identity and canceling terms.

Eq. (53) provides information on how well the series of Eq. (51) approximates $f(\theta)$, for the error term is $\sec(\theta)\cos(2n\theta)$. If we integrate this term multiplied by a test function $\phi(\theta)$, i.e., $\int_0^{2\pi} \phi(\theta)\sec(\theta)\cos(2n\theta)\,\mathrm{d}\theta$, and apply the Riemann–Lebesgue lemma [21,22], it is an easy exercise to show that this integral converges to zero as $n \to \infty$ provided that $\phi(\theta)\sec(\theta)$ is continuous and this requires that $\phi(\theta)$ be zero at $\theta = \pi/2$ and $3\pi/2$. For such $\phi(\theta)$, then

$$\int_0^{2\pi} \phi(\theta)\sec(\theta)\,\mathrm{d}\theta = \lim_{n\to\infty} \int_0^{2\pi} \phi(\theta) \sum_{k=1}^{n} 2(-1)^k \cos((2k-1)\theta)\,\mathrm{d}\theta \tag{54}$$

and, therefore, the series of Eq. (51) converges to $f(\theta)$ in this weak sense [22,23]. In addition, the divergent series of Eq. (51) can be shown to be Abel summable [23].

4. Comparisons with other methods and conclusions

The results presented in the two formulations above clearly indicate that the range of applicability of the artificial parameter Linstedt–Poincaré method for the singular equation considered here depends on how the equation is written. This technique is not iterative and, at first order, it provides identical results to the first iteration of Mickens' two-level iterative method (cf. Eqs. (7), (24) and (35)) who considered Eq. (2) rather than Eq. (1); at first-order, it also provides a more accurate frequency of oscillation than a first-order harmonic balance technique applied to Eq. (2) (cf. Eqs. (4), (24) and (35)). However, a second-order harmonic balance approximation (cf. Eq. (5)) predicts a more accurate frequency of oscillation than the two implementations of the artificial parameter Linstedt–Poincaré method presented here, if only two terms in Eq. (20) are considered and the harmonic balance procedure is applied to Eq. (2).

Mickens' two-level iterative method provides approximations to both the solution and the frequency of oscillation in an iterative/sequential manner as indicated in Eq. (6) and this iterative procedure can be shown to converge because the right-hand side of Eq. (6) is a Lipschitz continuous function of its arguments [24–26]. On the other hand, the artificial parameter Linstedt–Poincaré method presented here provides both the solution and the frequency of oscillation as power series of the artificial parameter (cf. Eqs. (19) and (20)) and this parameter is set to unity at the end of the calculations. As indicated above, the series of Eq. (16) is only valid for small amplitudes, whereas that of Eq. (32) has been shown to be valid for large amplitudes by means of Mathematica. Moreover, as indicated in the paragraph following Eq. (32), the convergence of the series for the frequency of oscillation is not a monotonic function of the number of terms of the series of Eq. (20).

In a recent paper, Beléndez et al. [17] determined the frequency of oscillation of Eq. (1) by means of a linearized harmonic balance procedure [27–29] applied directly to that equation. For the lowest harmonic balance approximation, these authors determined a frequency of oscillation identical to that of Eq. (4). By expressing the second approximation to the solution as the sum of the first approximation, i.e., $A\cos(\omega_0 t)$, where $\omega_0^2 = 2/A^2$, which satisfies the initial conditions and an unknown function $y_1(t)$, substituting this approximation into Eq. (1), linearizing the resulting expression, approximating $y_1(t)$ by means of trigonometric functions that satisfy homogeneous initial conditions, and applying the method of harmonic balance to the resulting equation, they obtained the following approximation to the frequency of oscillation:

$$\omega_{WL2} = \frac{1.2193273}{A}, \tag{55}$$

which differs from the exact one by 2.71%. However, by applying an exact second-order harmonic balance procedure, Beléndez et al. [17] obtained

$$\omega_{SHB2} = \frac{1.23733085}{A}, \tag{56}$$

which differs from the exact one by 1.275% and is, therefore, a better approximation to the frequency of oscillation than those of Mickens' second-order harmonic balance procedure (cf. Eq. (5)) applied to Eq. (2), the first iteration of Mickens' two-level iterative technique, i.e., Eq. (7), and the two-term approximations provided by the artificial parameter Linstedt–Poincaré method presented in this paper, i.e., Eqs. (29) and (41). A summary of the frequencies of oscillation of Eq. (1) and their errors obtained with different methods is presented in Table 1; these tables also include the equation to which the methods have been applied.

A comparison between the artificial parameter Linstedt–Poincaré method presented here and the linearized harmonic balance employed by Beléndez et al. [17] indicates that the former provides approximation to both the solution and the frequency of oscillations as power series of the artificial parameter (cf. Eqs. (19) and (20)) which is set to unity at the end of the calculations, whereas the latter provides successive approximations to both the solution and the frequency of oscillations. In both methods, the equations governing the second- and higher-order approximations are linear (except for the exact second-order harmonic balance procedure,

Table 1
Frequencies of oscillation (ω_M) and relative errors (E_M) in the frequency of oscillation of method M

ω_{HB1} (Eq.)	ω_{HB2} (Eq.)	ω_{LHB1} (Eq.)	ω_{LHB2} (Eq.)	ω_{LHB2E} (Eq.)
$\frac{1.4142}{A}$ (2)	$\frac{1.2728}{A}$ (2)	$\frac{1.4142}{A}$ (1)	$\frac{1.2193}{A}$ (1)	$\frac{1.1547}{A}$ (16)
ω_{MLP1} (Eq.)	ω_{MLP2} (Eq.)	ω_{MLP1} (Eq.)	ω_{MLP2} (Eq.)	ω_{MLP10} (Eq.)
$\frac{1.2373}{A}$ (1)	$\frac{1.1547}{A}\sqrt{1+\frac{5A^2}{96}}$ (16)	$\frac{1.1547}{A}$ (32)	$\frac{1.0453}{A}$ (32)	$\frac{1.2483}{A}$ (32)
ω_{MLP11} (Eq.)	ω_{IT1} (Eq.)	ω_{HPM} (Eq.)	ω_{HPM} (Eq.)	ω_{HPM} (Eq.)
$\frac{1.2496}{A}$ (32)	$\frac{1.1547}{A}$ (6)	$\frac{1.4142}{A}$ (8)	$\frac{1.1547}{A}$ (9)	$\frac{1.1547}{A}$ (10)
E_{HB1} (Eq.)	E_{HB2} (Eq.)	E_{LHB1} (Eq.)	E_{LHB2} (Eq.)	E_{LHB2E} (Eq.)
12.8 (2)	1.6 (2)	12.8 (1)	2.7 (1)	1.3 (1)
E_{MLP1} (Eq.)	E_{MLP1} (Eq.)	E_{MLP2} (Eq.)	E_{MLP10} (Eq.)	E_{MLP11} (Eq.)
7.9 (16)	7.9 (32)	16.6 (32)	0.04 (32)	0.03 (32)
E_{IT1} (Eq.)	E_{HPM} (Eq.)	E_{HPM} (Eq.)	E_{HPM} (Eq.)	
7.9 (32)	12.8 (8)	7.9 (9)	7.9 (10)	

(M = HB1, HB2, LHB1, LHB2, LHB2E, MLP1, MLP2, MLP10, MLP11, HPM and IT1 correspond to the first- and second-order harmonic balance procedures [1], the linearized first- and second-order harmonic balance methods [17], the linearized second-order harmonic balance technique with exact second-order harmonic balance [17], the artificial parameter Linstedt–Poincaré method presented in this paper with one, two, ten and eleven terms, the first-order homotopy perturbation method and the first iteration of Mickens' two-level iterative technique [1], respectively. The exact frequency of oscillation is $\omega_{\mathrm{ex}} = 1.2533/A$ and the relative error of method M is determined as $E = 100 \times |(\omega_{\mathrm{ex}} - \omega_M)/\omega_{\mathrm{ex}}|$. The error of MLP2 for Eq. (29) (cf. Eq. (16)) is $E_{\mathrm{MLP2}} = 100 \times |1 - 0.9213\sqrt{1 + 5A^2/96}|$, depends on the amplitude of motion and is not reported in the table.

i.e., Eq. (36), of Beléndez et al. [17]) and satisfy homogeneous initial conditions; the artificial parameter Linstedt–Poincaré method makes use of non-secularity conditions at each order in the expansion in terms of the artificial parameter, whereas the linearized harmonic balance technique employs harmonic balance. Moreover, Eq. (50) is exactly the same equation as Eq. (25) of Beléndez et al. [17] when written in terms of t. However, in the artificial parameter Linstedt–Poincaré method presented here, Eq. (50) would be integrated exactly as it was done with Eqs. (23), (26), (34) and (37) above, whereas Beléndez et al. [17] used an approximation (cf. their Eq. (28)) to solve their Eq. (25) and this approximation is the one that gave Eq. (53) and, as these authors stated, it exhibits opposite results to those expected. For this reason, these authors proposed an exact second-order harmonic balance procedure that makes use of their nonlinear Eq. (36) for the correction $u(t)$ which is then approximated by a two-term harmonic approximation that satisfies homogeneous initial conditions. The third approximation in the linearized harmonic balance method is determined from a perturbation to the second approximation and has an analogous expression to Eq. (50) but with y_0 and y_1 replaced by y_1 and y_2, respectively, in Eq. (50), whereas the third-order approximation of the method presented here is governed by Eq. (52).

Perhaps the largest difference between these two techniques comes from the fact that the artificial parameter Linstedt–Poincaré method is applied to Eqs. (16) and (32) rather than to Eq. (1), whereas the linearized harmonic balance procedure is applied directly to Eq. (1) and provides an approximation to the solution that it is independent of the amplitude. In addition, the frequency of oscillation is determined by the series of Eq. (20) in the method presented here, whereas it is improved in a sequential manner in the linearized harmonic balance procedure. On the other hand, the artificial parameter Linstedt–Poincaré method when applied to Eq. (16) provides a solution that it is only valid for small amplitudes, whereas, when applied to Eq. (32), it provides a solution that is valid for small and large amplitudes as has been verified with Mathematica. Furthermore, the linearized harmonic balance method when applied directly to Eq. (1) provides an approximation to the frequency of oscillation at second order which is more accurate than that provided by a two-term approximation of the artificial parameter Linstedt–Poincaré method when applied to Eq. (32).

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